On the Monotonicity of the Speed of Random Walks on a Percolation Cluster of Trees*

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Abstract

We consider the simple random walk on the infinite cluster of the Bernoulli bond percolation of trees, and investigate the relation between the speed of the simple random walk and the retaining probability p by studying three classes of trees. A sufficient condition is established for Galton-Watson trees.

1 Introduction

The simple random walk $\{X_n\}$ on graph $\mathcal{G} = (V, E)$ is defined as a Markov chain on the set V of vertices with transition probability $p(x, y) = 1/d_x$ if $x, y \in V$ and there is an edge between them. The degree d_x of vertex x is the number of edges connecting x to other vertices.

Let o be a fixed vertex and $|x|$ the graphic distance between o and x, i.e., the minimum number of edges in a path from o to x. Suppose the simple random walk starts from o, i.e., $X_0 = o$. We call $\lim_n |X_n|/n$, if it exists, the speed of the simple random walk. For the simple random walk in Z^d, the speed is zero. On the other hand the speed is $(d-1)/(d+1)$ for the simple random walk on the regular tree T_d. By regular tree T_d, we mean a tree that the degree of every vertex except the root is $d+1$. The degree of the root o is d.

A non-trivial example is the simple random walk on a Galton-Watson tree. Let $\{p_n, n = 0, 1, 2, \cdots\}$ be the offspring distribution of a Galton-Watson process. Each

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individual produces offsprings independently according the same law. There is a rooted tree for each realization, and the correspondence induces a probability measure in the set of rooted trees. A rooted tree, drawn according to this measure, is called a Galton-Watson tree. It is shown in [5] that the speed on a Galton-Watson is a constant a.s.. More remarkably, an explicit formula is given when $p_0 = 0$.

$$\text{Speed } = \sum_{k=1}^{\infty} p_k \frac{k-1}{k+1}. \tag{1}$$

In *p-Bernoulli bond percolation* in $\mathcal{G}$, each edge of $\mathcal{G}$ is independently declared *open* with probability p and *closed* with probability $1-p$. Thus a bond percolation ω is a random subset of E. We usually identify the percolation ω with the subgraph of $\mathcal{G}$ consisting of all open edges and their end-vertices. A connected component of this subgraph is called an *open cluster*, or simply a *cluster*. Let $p_c = p_c(\mathcal{G}) = \inf\{p$: there is an infinite cluster $a.s.\}$. By coupling, the infinite cluster is increasing in p. The probability that there is an infinite cluster is monotone in p. When $p \in (p_c, 1)$, with positive probability the open cluster $\mathcal{C}$ that contains o is infinite.

The simple random walk on the infinite cluster of Z^d is first studied in [3]. A more systematic investigation is initiated in [1]. We like to investigate the relation between the speed of the simple random walk on an infinite cluster and the retaining probability p by observing three examples: the regular tree, Galton-Watson trees, and the binary tree with pipes, which is obtained by adding a pipe to each vertex of the binary tree. See Figure 1.

Proposition 1 *The speed of the simple random walk on an infinite cluster of a regular tree is increasing in p.*

The main idea of the proof is to use formula (1) and the decomposition of an infinite Galton-Watson tree as a *backbone* and *bushes*. As p increases, the backbone gets larger, the mean size and the number of bushes get smaller. It is tempting to think that the monotone relation holds for a large class of graphs.

Question A: *Is the speed of the simple random walk on an infinite cluster of a transitive graph increasing in p?*

We believe the answer is "yes". However, this statement can not be pushed any further, as we see in the following example of the binary tree with pipes. The binary tree is chosen because the extinction probability can be explicitly calculated. Assume that $p > 1/2$ and the cluster containing the root o is infinite. In addition to the backbone and bushes, there are also pipes of random length. The distribution of the length is geometric and the mean length of a pipe is $p/(1-p)$. Therefore, the larger the p is, the longer a pipe is, and the longer the time for the excursion on a pipe is. As p increases, there is a competition between the increase of time spent on the pipe and the decrease of time spent on the bush.

Example 2 *The speed of the simple random walk on an infinite cluster of the binary tree with pipes is*

$$\frac{1}{3}\,\frac{(2p-1)^2}{p^2+(1-p)^2}\;\frac{1-p}{\left(2p^3-6p^2+3p+3\right)}$$

which is not monotone in p.

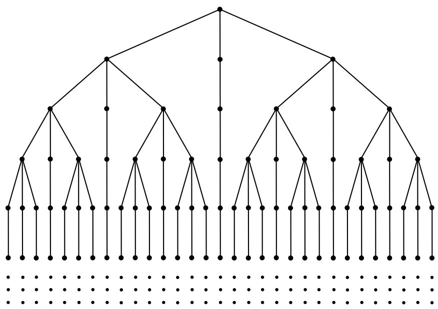

Figure 1: The binary with pipes

Very often a random environment (or graph) exhibits nice characters of a homogeneous environment (or a transitive graph). So we take the Galton-Watson trees. Let $\{p_k\}$ be the offspring distribution, and $m_1 = \sum_k kp_k$ the mean number of offsprings. Suppose the retaining probability $p > 1/m_1$ and the open cluster $\mathcal{C}_o$ of p-Bernoulli bond percolation containing the root is infinite. Run the simple random walk starting from the root.

Theorem 3 *Let $f(s) = \sum_k p_k s^k$ be the generating function and $f'(s)$ the derivative of f. Suppose that*

$$\frac{(1-s)f'(s)}{1-f(s)} \text{ is increasing in } s \tag{2}$$

for $s \in (1/m_1, 1)$. Then the speed of the simple random walk on an infinite cluster of a Galton-Watson tree is an increasing function of p. Furthermore it is continuous for $p \in [1/m_1, 1]$ and differentiable for $p \in (1/m_1, 1)$.

It is not difficult to verify (2) in general, as we now examine some special cases.

Example 4 *The geometric distribution, i.e.* $p_k = a^k(1-a)$ *for* $0 < a < 1$. Then

$$f(s) = \frac{1-a}{1-as}, \qquad \text{and} \quad \frac{(1-s)f'}{1-f} = \frac{1-a}{1-as}$$

which is clearly increasing in s. With a little extra work one can show that the *negative binomial distribution* also satisfies (2).

Example 5 *The Poisson distribution with parameter* μ. In this case, $f(s) = e^{s\mu-\mu}$ and $f'(s) = \mu f(s)$. Then the derivative of $(1-s)f'/(1-f)$ is $\phi(s)f/(1-f)^2$ where $\phi(s) = \mu(1-s) - 1 + f$. Since $\phi(1)$=0, and $\phi'(s) < 0$ for $s < 1$, so $\phi(s) \geq 0$ for $s \leq 1$.

Example 6 *Binomial distribution* $B(n,p)$. In this case, $f = (1-p+sp)^n$ and we can similarly verify that the derivative of $(1-s)f'/(1-f)$ is non-negative.

Proof of Proposition 1. The regular tree is a Galton-Watson tree with the degenerated distribution: $p_d = 1$. In this case $f(s) = s^d$, $f'(s) = ds^{d-1}$ and

$$\frac{(1-s)f'}{1-f} = d\,\frac{s^{d-1}-s^d}{1-s^d}$$

is indeed increasing for $s \in (0,1)$. The condition of Theorem 3 is satisfied and Proposition 1 is proved. □

There is a close relation between the speed of the simple random walk on a graph and the anchored expansion constant of the graph. It is shown in [6] that a positive anchored expansion constant implies that the speed is positive. We also know from [2] that the anchored expansion constant of the infinite cluster is positive if the anchored expansion constant of the original graph is positive and if the retaining probability of the Bernoulli bond percolation is close to 1. In light of the previous discussion we may ask the following question.

Question B: *Is the anchored expansion constant of an infinite cluster of a transitive graph increasing in* p*? Is it continuous in* p*?*

Although a negative answer would be very surprising, we even do not know the answer for a regular tree. In this case the infinite cluster is just a Galton-Watson tree and the anchored expansion constant is a constant $a.s.$ for all Galton-Watson trees. It is positive and is less than $m-1$, where m is the mean number of offsprings [2]. A moment of thought reveals that the inequality could be strict.

2 Galton-Watson Trees

Take a Galton-Watson tree with offspring distribution $\{p_k, k = 0, 1, 2, \cdots\}$ and consider the Bernoulli bond percolation on the tree with retaining probability p. The open cluster containing the root o is again a Galton-Watson tree with offspring distribution

$$\bar{p}_l = \sum_{r=0}^{\infty} p_{l+r} p^l (1-p)^r C_{l+r}^r, \qquad \text{for } l = 0, 1, 2, \cdots .$$

In order to have an infinite cluster, the retaining probability $p > 1/m_1$ where $m_1 = \sum_k k p_k$ is the mean number of offsprings.

We are now working with two Galton-Watson processes: $\{p_k\}$ and $\{\bar{p}_k\}$. The extinction probability is denoted by q and ρ respectively. The mean of the former is denoted by m_1. The generating function $f(s) = \sum_k p_k s^k$ is exclusively for the former.

Lemma 2.1 *ρ is monotone and differentiable in p for $p \in (1/m_1, 1)$. Let $\lambda = 1 - p + p\rho$. Then*

$$\frac{d\rho}{dp} = -\frac{(1-\rho) f'(\lambda)}{1 - p f'(\lambda)}. \tag{3}$$

Proof. By definition ρ satisfies the following equation.

$$\rho = \sum_{k=0}^{\infty} \bar{p}_k \rho^k = \sum_{l=0}^{\infty} p_l (1 - p + p\rho)^l. \tag{4}$$

Equivalently,

$$\frac{\lambda - 1 + p}{p} = \rho = f(\lambda), \tag{5}$$

or

$$p = \frac{1-\lambda}{1 - f(\lambda)}. \tag{6}$$

So p is continuous and differentiable in λ. Except the degenerated case that $f(s) = s$, $dp/d\lambda < 0$ by the convexity of f. Therefore λ as the inverse function is also continuous and differentiable in p. Take the derivative of the both sides of (6) with respect to p we get,

$$\frac{d\lambda}{dp} = -\frac{1-f(\lambda)}{1-pf'(\lambda)}.$$

This together with (5) implies (3). As a by-product, $1 - pf'(\lambda) > 0$. □

We assume that the open cluster $\mathcal{C}_o$ containing the root is infinite. As a Galton-Watson tree, $\mathcal{C}_o$ can be constructed as follows, see [4]. Begin with the root which is declared to be *green*. Add to the root a random number of edges according to probability distribution $P(Y = k) = \bar{p}_k(1-\rho^k)/(1-\rho)$. Declare the other end vertex of newly-added edge *green* with probability $1-\rho$ and *red* with probability ρ, independent of each other. If all the newly added vertices are red, discard the entire assignment and reassign *green/red* all over again. For each *green* vertex, repeat the same procedure. For each *red* vertex, attach to it independently a random number of red vertices according to the distribution

$$\hat{p}_k = \bar{p}_k\rho^{k-1} \quad \text{for } k \geq 0. \tag{7}$$

The infinite tree consisting of green vertices will be called the *backbone* and a connected component of red vertices is called a *bush*. The backbone is a Galton-Watson tree generated according to the distribution:

$$\tilde{p}_l = \sum_{r=0}^{\infty} \bar{p}_{l+r}\rho^r(1-\rho)^{l-1}C_{l+r}^r \quad \text{for } l \geq 1. \tag{8}$$

A bush is a subcritical Galton-Watson tree with offspring distribution given by (7). Let M be the mean size of a bush.

Lemma 2.2

$$M = \frac{1}{1-pf'(\lambda)}.$$

Proof. Let $\hat{m}$ be the mean number of offsprings. Then, by (7),

$$\begin{aligned}
\hat{m} &= \sum_{k=0}^{\infty} \hat{p}_k k = \sum_{k=0}^{\infty} \bar{p}_k \rho^{k-1} k \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} p_{k+l} p^k (1-p)^l C_{k+l}^l \rho^{k-1} k \\
&= \sum_{n=0}^{\infty} \frac{p_n}{\rho} \lambda^n \sum_{k+l=n} (\frac{p\rho}{\lambda})^k (\frac{1-p}{\lambda})^l C_{k+l}^l k \\
&= \sum_{n=0}^{\infty} \frac{p_n}{\rho} \lambda^n (\frac{p\rho}{\lambda}) n \\
&= p \sum_{n=0}^{\infty} p_n \, n \lambda^{n-1} \\
&= p f'(\lambda).
\end{aligned}$$

Hence

$$M = \sum_{k=0}^{\infty} \hat{m}^k = \frac{1}{1-\hat{m}}.$$

□

Lemma 2.3 *The speed of the simple random walk on the backbone is increasing in p if $(1-s)f'(s)/(1-f(s))$ is increasing in s.*

Proof. Let $S(p)$ be the speed of the simple random walk on the backbone. By formula (1),

$$\begin{aligned}
S(p) &= \sum_{k=1}^{\infty} \tilde{p}_k \frac{k-1}{k+1} = 1 - 2\sum_k \tilde{p}_k \frac{1}{k+1} \\
&= 1 - 2\sum_{k=1}^{\infty}\sum_{r=0}^{\infty} \bar{p}_{k+r}\rho^r(1-\rho)^{k-1}C_{k+r}^r \frac{1}{k+1} \\
&= 1 - 2\sum_{l=1}^{\infty} \bar{p}_l \sum_{k+r=l,k\geq 1} \rho^r(1-\rho)^{k-1}\frac{l!}{r!(k+1)!} \\
&= 1 - \frac{2}{(1-\rho)^2}\sum_{l=1}^{\infty}\frac{\bar{p}_l}{l+1}\left[1-\rho^{l+1}-\rho^l(l+1)(1-\rho)\right] \\
&= 1 - \frac{2}{(1-\rho)^2}\sum_{l=0}^{\infty}\frac{\bar{p}_l}{l+1}\left[1-\rho^{l+1}-\rho^l(l+1)(1-\rho)\right] \\
&= 1 + \frac{2}{1-\rho}\sum_{l=0}^{\infty}\bar{p}_l\rho^l - \frac{2}{(1-\rho)^2}\sum_{l=0}^{\infty}\bar{p}_l\frac{1-\rho^{l+1}}{l+1} \\
&= 1 + \frac{2\rho}{1-\rho} - \frac{2}{(1-\rho)^2}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} p_{l+r}p^l(1-p)^r C_{l+r}^r \frac{1-\rho^{l+1}}{l+1} \\
&= \frac{1+\rho}{1-\rho} - \frac{2}{(1-\rho)^2 p}\sum_{n=0}^{\infty}\frac{p_n}{n+1}\sum_{l+r=n}\frac{(n+1)!}{r!(l+1)!}p^{l+1}(1-p)^r(1-\rho^{l+1}) \\
&= \frac{1+\rho}{1-\rho} - \frac{2}{(1-\rho)^2 p}\sum_{n=0}^{\infty}\frac{p_n}{n+1}[1-(1-p+p\rho)^{n+1}] \\
&= \frac{1+\rho}{1-\rho} - \frac{2}{(1-\rho)^2 p}\sum_{n=0}^{\infty}p_n\frac{1-\lambda^{n+1}}{n+1}.
\end{aligned}$$

Take the derivative of $S(p)$ with respect to p.

$$\begin{aligned}
S'(p) &= \frac{2\rho'}{(1-\rho)^2} + \frac{2}{(1-\rho)^2 p}\sum_{n=0}^{\infty}p_n\lambda^n\lambda' - 2\frac{2p\rho'-(1-\rho)}{(1-\rho)^3p^2}\sum_{n=0}^{\infty}p_n\frac{1-\lambda^{n+1}}{n+1} \\
&= \frac{2}{(1-\rho)^2}\left[\rho' + \frac{\rho}{p}(-1+\rho+p\rho') - \frac{2p\rho'-(1-\rho)}{(1-\rho)p^2}\sum_{n=0}^{\infty}p_n\frac{1-\lambda^{n+1}}{n+1}\right].
\end{aligned}$$

Plugging (3),

$$\begin{aligned}
S'(p) &= \frac{2}{(1-\rho)^2}\left[-(1+\rho)\frac{(1-\rho)f'}{1-pf'} - \frac{\rho(1-\rho)}{p} + \frac{1+pf'}{(1-pf')p^2}\sum_{n=0}^{\infty}p_n\frac{1-\lambda^{n+1}}{n+1}\right] \\
&= \frac{2\Psi(p)}{(1-\rho)^2(1-pf')p^2}
\end{aligned}$$

where

$$\Psi(p) = -(1-\rho)p(\rho+pf') + (1+pf')\sum_n p_n(1-\lambda^{n+1})/(n+1).$$

Furthermore, $\Psi(1/m_1) = 0$ and

$$\Psi'(p) = \left[p(1-\rho) - \sum_n p_n \frac{1-\lambda^{n+1}}{n+1} \right] (-f' - pf''\lambda').$$

It is enough to show $\Psi'(p) \geq 0$ for $p \in (1/m_1, 1)$. This can be done in two parts. First, it is easy to show that

$$p(1-\rho) - \sum_n p_n \frac{1-\lambda^{n+1}}{n+1} \geq 0,$$

since its value at $p = 1/m_1$ is 0 and its derivative with respect to p is $(1-\rho)^2/(1 - pf'(\lambda)) \geq 0$. Secondly,

$$-f' - pf''\lambda' = -\frac{d}{dp}(pf'(\lambda)) = \frac{d}{d\lambda}\left(\frac{(1-\lambda)f'(\lambda)}{1-f(\lambda)}\right)\left(-\frac{d\lambda}{dp}\right) \geq 0$$

by the assumption that $(1-s)f'(s)/(1-f(s))$ is increasing in s. $\square$

A trajectory of the simple random walk on $\mathcal{C}_o$ is a sequence of red vertices and green vertices. A consecutive sequence of red vertices, together with the two green vertices immediately before and after the sequence, is called an *excursion*. If the end of one excursion is the start of another excursion in a trajectory, we say two excursions are consecutive. Let $N(p,k)$ be the conditional expectation of the number of consecutive excursions into bushes before moving to an adjacent vertex of x in the backbone, given the degree of x in the backbone is $k+1$.

Lemma 2.4

$$N(p,k) = \frac{p\rho}{(k+1)} \frac{f^{(k+1)}(\lambda)}{f^{(k)}(\lambda)}.$$

Here $f^{(k)}(\lambda)$ is the k-th derivative of f.

Proof. Suppose that the degree of vertex x in the backbone is $k+1$ and that there are l bushes attached to x. In this case the mean number of consecutive excursions into bushes before moving to an adjacent vertex of the backbone is $l/(k+1)$.

The probability that there are l bushes attached to a vertex of degree $k+1$ in the backbone is $\bar{p}_{k+l}(1-\rho)^{k-1}\rho^l C^l_{k+l}$. Here we have implicitly assumed that the vertex is not the root, since the simple random walk in a Galton-Watson tree is transient. Take the conditional expectation of the number of consecutive excursions into bushes.

$$\begin{aligned}
N(p,k) &= \frac{\sum_{l=0}^{\infty}\frac{l}{k+1}\bar{p}_{k+l}(1-\rho)^{k-1}\rho^{l}C_{k+l}^{l}}{\sum_{l=0}^{\infty}\bar{p}_{k+l}(1-\rho)^{k-1}\rho^{l}C_{k+l}^{l}} \\
&= \frac{\sum_{l=0}^{\infty}\frac{l}{k+1}\sum_{r=0}^{\infty}p_{k+l+r}p^{k+l}(1-p)^{r}C_{k+l+r}^{r}(1-\rho)^{k}\rho^{l}C_{k+l}^{l}}{\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}p_{k+l+r}p^{k+l}(1-p)^{r}C_{k+l+r}^{r}(1-\rho)^{k}\rho^{l}C_{k+l}^{l}} \\
&= \frac{\sum_{l=1}^{\infty}\sum_{r=0}^{\infty}\frac{(k+l+r)!}{(k+1)!r!(l-1)!}p_{k+l+r}p^{l}(1-p)^{r}\rho^{l}}{\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k+l+r)!}{k!r!l!}p_{k+l+r}p^{l}(1-p)^{r}\rho^{l}} \\
&= \frac{\sum_{s=1}^{\infty}p_{k+s}\frac{(k+s)!}{(k+1)!(s-1)!}p\rho(1-p+p\rho)^{s-1}}{\sum_{s=0}^{\infty}p_{k+s}\frac{(k+s)!}{k!s!}(1-p+p\rho)^{s}} \\
&= \frac{p\rho\sum_{s=1}^{\infty}p_{k+s}C_{k+s}^{k+1}\lambda^{s-1}}{\sum_{s=0}^{\infty}p_{k+s}C_{k+s}^{k}\lambda^{s}} \\
&= \frac{p\rho}{(k+1)}\frac{f^{(k+1)}(\lambda)}{f^{(k)}(\lambda)}.
\end{aligned}$$

$\square$

Proof of Theorem 3. While the infinite cluster $\mathcal{C}_o$ is decomposed as the *backbone* and *bushes*, the simple random walk on $\mathcal{C}_o$ is also decomposed as the simple random walk on the backbone and excursions into bushes. Thus the simple random walk on $\mathcal{C}_o$ can be regarded as a delayed simple random walk on the backbone with delays caused by excursions into bushes.

The average time spent in an excursion into a bush, by the ergodic theorem of Markov chains, is 2 times the size of the bush (the number of red vertices). Thus the average of random delays at a vertex of degree $k+1$ is

$$2\ M\ \ N(p,k) = \frac{2}{1-\hat{m}}\ \frac{p\rho}{(k+1)}\ \frac{f^{(k+1)}(\lambda)}{f^{(k)}(\lambda)}.$$

According to [5], the frequency that the simple random walk on the backbone visits a vertex of degree $k+1$ is $\tilde{p}_k$ defined in (8). Notice that

$$\tilde{p}_k = \frac{f^{(k)}(\lambda)}{k!}p^{k}(1-\rho)^{k-1}. \tag{9}$$

Then the average of random delays at a vertex is as follows.

$$\begin{aligned}
\sum_{k=1}^{\infty}\tilde{p}_k 2MN(k) &= \frac{2p\rho}{1-\hat{m}}\sum_{k=1}^{\infty}\tilde{p}_k\frac{1}{k+1}\frac{f^{(k+1)}}{f^{(k)}} \\
&= \frac{2p\rho}{1-pf'}\sum_{k=1}^{\infty}\frac{f^{(k)}}{k!}\cdot p^k(1-\rho)^{k-1}\frac{1}{k+1}\frac{f^{(k+1)}}{f^{(k)}} \\
&= \frac{2p\rho}{1-pf'}\sum_{k=1}^{\infty}\frac{f^{(k+1)}}{(k+1)!}\cdot p^k(1-\rho)^{k-1} \\
&= \frac{2p\rho}{1-pf'}\frac{1}{p(1-\rho)}\sum_{k=2}^{\infty}\frac{f^{(k)}}{k!}\cdot p^k(1-\rho)^{k-1} \\
&= \frac{2p\rho}{1-pf'}\frac{1}{p(1-\rho)}\sum_{k=2}^{\infty}\tilde{p}_k \\
&= \frac{2p\rho}{1-pf'}\frac{1}{p(1-\rho)}(1-\tilde{p}_1) \\
&= \frac{2p\rho}{1-pf'}\frac{1}{p(1-\rho)}(1-f'p) \\
&= \frac{2\rho}{1-\rho}.
\end{aligned}$$

The speed of the simple random walk on $\mathcal{C}_o$, by the ergodic theorem of Markov chains, is

$$\left(1+\frac{2\rho}{1-\rho}\right)^{-1}S(p)=\frac{1-\rho}{1+\rho}\sum_{k=1}^{\infty}\tilde{p}_k\frac{k-1}{k+1}. \tag{10}$$

It is monotone by Lemmas 2.1 and 2.3. Its continuity and differentiability follow from Lemma 2.1, (9) and (10). □

Remark: The speed of the simple random walk on the infinite cluster $\mathcal{C}_o$ can be calculated directly by applying another formula from [5]. In our notation,

$$\text{Speed } =\sum_{k=0}^{\infty}\bar{p}_k\frac{k-1}{k+1}\,\frac{1-\rho^{k+1}}{1-\rho^2}. \tag{11}$$

But a compact form blurs the monotone relations, as we see in the following calculation.

$$\text{Speed } = 1-\frac{2}{(1-\rho^2)p}\sum_{n=0}^{\infty}p_n\frac{1-\lambda^{n+1}}{n+1}$$

$$= 1-\frac{2}{(1-\rho^2)p}\sum_{n=0}^{\infty}\frac{p_n}{n+1}+\frac{2}{(1-\rho^2)p}\sum_{n=0}^{\infty}p_n\frac{\lambda^{n+1}}{n+1}.$$

The first two terms are increasing in p. A sufficient condition for the monotonicity of the speed would be that

$$\rho \geq \frac{2p\rho f'(\lambda) + (1+\rho)(1-pf'(\lambda))}{(1-\rho^2)p} \sum_{n=0}^{\infty} p_n \frac{\lambda^{n+1}}{n+1}.$$

The difficulty lies in that we know little of ρ. Even for a regular tree, we only have a good estimate of ρ.

$$\rho(p) \leq \frac{1-p}{d^2p^2-p} \quad \text{for} \quad p \in (1/d, 1).$$

An exceptional case is the binary tree for which $\rho = (1-p)^2/p^2$ can be found explicitly. This is used in Example 2.

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