Anchored Expansion, Percolation and Speed

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Abstract. The anchored expansion of a graph G is the infimum over all finite connected vertex sets S that contain a fixed basepoint, of the boundary-to-volume ratio $|\partial S|/|S|$. We solve several problems raised in a paper by Benjamini, Lyons and Schramm (1999) where this notion was introduced. We prove that the positivity of the anchored expansion is preserved under percolation with parameter p sufficiently close to 1, and also under a random stretch when the stretching law has an exponential tail. The importance of anchored expansion was exhibited by Virág (2000), who showed that positivity of anchored expansion implies that simple random walk on G has positive speed. We also study simple random walk in the infinite cluster of p-Bernoulli bond percolation in Cayley graphs of amenable groups of exponential growth known as "lamplighter groups". We prove that at least for large p, the speed of random walk on the infinite cluster is positive if and only if the speed is positive for random walk on the whole group.

§1. Introduction.

Denote by $V(\mathbb{G})$ and $E(\mathbb{G})$, respectively, the sets of vertices and edges of an infinite graph $\mathbb{G}$. For $S \subset V(\mathbb{G})$, denote by $|S|$ the cardinality of S and by $|\partial S|$ the number of edges that have one end in S and the other in S^c. Say that S is *connected* if the induced subgraph on S is connected. Fix $o \in V(\mathbb{G})$. The **anchored expansion** of $\mathbb{G}$,

$$\iota_E^*(\mathbb{G}) := \lim_{n\to\infty} \inf \left\{ \frac{|\partial S|}{|S|}\,;\ o \in S \subset V(\mathbb{G}),\ S \text{ is connected},\ n \le |S| < \infty \right\}$$

was defined in Benjamini, Lyons & Schramm (1999), abbreviated as BLS (1999) thereafter. The quantity $\iota_E^*(\mathbb{G})$ is independent of the choice of basepoint o. It is related to the isoperimetric constant

$$\iota_E(\mathbb{G}) := \inf \left\{ \frac{|\partial S|}{|S|}\,;\ S \subset V(\mathbb{G}),\ S \text{ is connected},\ 1 \le |S| < \infty \right\}$$

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but is more “robust”. BLS (1999) asked if the positivity of $\iota_E^*(\mathbb{G})$ is preserved when $\mathbb{G}$ undergoes a random perturbation. It is easy to see that the positivity of $\iota_E(\mathbb{G})$ is lost in the following two random perturbations.

In p-**Bernoulli bond percolation** in $\mathbb{G}$, each edge of $\mathbb{G}$ is independently declared *open* with probability p and *closed* with probability $1-p$. Thus a bond percolation ω is a random subset of $E(\mathbb{G})$. We usually identify percolation ω with the subgraph of $\mathbb{G}$ consisting of all *open* edges and their end vertices. A connected component of this subgraph is called an *open cluster*, or simply a *cluster*. The probability that there is an infinite cluster is monotone in p. Let $p_c = \inf\{p\colon$ there is an infinite cluster *a.s.*$\}$. When $p > p_c$, with positive probability the open cluster containing o is infinite. The proof of Theorem 2 of Benjamini and Schramm (1996) actually proves that $p_c \le 1/(\iota_E^*(\mathbb{G})+1)$.

THEOREM 1.1. *If $p > 1 - h/(1+h)^{1+\frac{1}{h}}$ for some $0 < h < \iota_E^*(\mathbb{G})$, then almost surely the open cluster $\mathbb{F}$ containing o is either finite or $\iota_E^*(\mathbb{F}) > 0$.*

Let $\mathbb{G}$ be an infinite graph of bounded degree and pick a probability distribution ν on the positive integers. Replace each edge $e \in E(\mathbb{G})$ by a path of length L_e, where L_e is distributed according to ν, and all L_e's are independent. Let $\mathbb{G}^\nu$ denote the random graph obtained in this way. We call $\mathbb{G}^\nu$ a **random stretch** of $\mathbb{G}$.

THEOREM 1.2. *Suppose that $\mathbb{G}$ is an infinite graph of bounded degree and $\iota_E^*(\mathbb{G}) > 0$. If ν has an exponential tail, then $\iota_E^*(\mathbb{G}^\nu) > 0$ a.s.*

On the other hand, if ν has a tail that does not decay exponentially, then there exists $\mathbb{G}$ such that $\iota_E^*(\mathbb{G}^\nu) = 0$ *a.s.* See Remark 2.2 in the next section. By *Galton-Watson tree* we mean a family tree of a Galton-Watson process.

COROLLARY 1.3. *$\iota_E^*(T) > 0$ a.s. for supercritical Galton-Watson trees given nonextinction.*

Theorem 1.2 and Corollary 1.3 answer Questions 6.3 and 6.4 of BLS (1999) while Theorem 1.1 partially answers Question 6.5 of the same paper.

For a vertex x, denote by $|x| = |x|_{\mathbb{G}}$ the distance (the least number of edges on a path) from x to the basepoint o in $\mathbb{G}$. When it exists, the limit $\lim_n |X_n|/n$ is called the *speed* of the simple random walk $\{X_n\}$ starting at o. Similarly, $\liminf |X_n|/n$ is called the lower speed of simple random walk $\{X_n\}$. The importance of anchored expansion is exhibited by the following theorem, conjectured in BLS (1999) (Conjecture 6.2) and proved in Virág (2000).

THEOREM 1.4. *Let $\mathbb{G}$ be a bounded degree graph with $\iota_E^*(\mathbb{G}) > 0$. Then $\liminf_{n\to\infty} |X_n|/n$ is positive a.s.*

Earlier, Thomassen (1992) showed that a condition weaker than $\iota_E^*(\mathbb{G}) > 0$ suffices for transience of the random walk $\{X_n\}$. Other applications of anchored expansion are in Häggström, Schonmann and Steif (2000).

In Section 3 we address another problem in BLS (1999) concerning the speed of simple random walk. Say that the graph $\mathbb{G}$ *grows sub-exponentially* if $\limsup |\{x \in V(\mathbb{G}) : |x| \leq n\}|^{1/n} = 1$. A graph $\mathbb{G}$ is *amenable* if there exists a sequence $\{S_n\}$ of finite subsets of $V(\mathbb{G})$ such that $\lim_n |\partial S_n|/|S_n| = 0$. In particular, a graph that grows sub-exponentially is amenable.

Simple random walk in a graph $\mathbb{G}$ has zero speed if $\mathbb{G}$ grows sub-exponentially. Consider again p-Bernoulli bond percolation in $\mathbb{G}$ with $p > p_c$. Theorem 1.3 of BLS (1999) states that if $\mathbb{G}$ is a non-amenable Cayley graph, then simple random walk in an infinite cluster of Bernoulli percolation on $\mathbb{G}$ has positive speed. It is therefore natural to study, as suggested in BLS (1999), a simple random walk in the infinite cluster of an amenable Cayley graph with exponential growth.

The lamplighter groups $\mathbb{G}_d$ are amenable groups with exponential growth, introduced by Kaimanovich and Vershik (1983). A vertex of $\mathbb{G}_d$ can be identified as $(m, \eta) \in \mathbb{Z}^d \times \{\text{finite subsets of } \mathbb{Z}^d\}$. Heuristically, $\mathbb{Z}^d$ is the set of lamps, η is the set of lamps which are on, and m is the position of the lamplighter, or "marker". In each step, either the lamplighter switches the current lamp (from on to off, or from off to on) or moves to one of the neighboring sites in $\mathbb{Z}^d$. Each vertex in $\mathbb{G}_d$ has degree $2d+1$; one edge corresponds to flipping the state of the lamp at location m, and the other $2d$ edges correspond to moving the marker. For example, if $d = 1$, the neighbors of (m, η) are $(m+1, \eta)$, $(m-1, \eta)$ and $(m, \eta\Delta\{m\})$, where $\eta\Delta\{m\}$ is $\eta \setminus \{m\}$ if $m \in \eta$, and is $\eta \cup \{m\}$ if $m \notin \eta$.

We now study simple random walk $\{X_n\}$ in the unique infinite cluster randomly generated by p-Bernoulli bond percolation in $\mathbb{G}_d$. Without loss of generality we may suppose the open cluster containing the basepoint o is infinite. If x is a vertex in this cluster, let $|x|_\omega$ be the distance in the cluster ω from x to o.

THEOREM 1.5. *The simple random walk in the infinite cluster of* $\mathbb{G}_1$ *or* $\mathbb{G}_2$ *has zero speed. Namely,* $\lim_n \frac{1}{n}|X_n|_\omega = 0$ *a.s.*

THEOREM 1.6. *Suppose that* $d \geq 3$. *If* $p > p_c(\mathbb{Z}^d)$, *then the simple random walk in the infinite cluster of Bernoulli bond percolation in* $\mathbb{G}_d$ *has positive speed. Namely,*

$$\lim_n \frac{|X_n|_\omega}{n} > 0 \quad a.s.$$

These results support Conjectures 1.4 and 1.5 of BLS (1999); they are extended in Theorems 3.1 and 3.2.

§2. Anchored Expansion.

The idea of the following lemma is from Kesten (1982).

LEMMA 2.1. *Let* $\mathcal{A}_n = \{S \subset V(\mathbb{G})\,;\; o \in S,\; S$ *is connected,* $|\partial S| = n\}$. *If* $\iota_E^*(\mathbb{G}) > h > 0$, *then for large* n

$$|\mathcal{A}_n| \le [\Psi(h)]^n$$

where

$$\Psi(h) = (1+h)^{1+\frac{1}{h}}/h.$$

Proof. Consider p-Bernoulli bond percolation in $\mathbb{G}$. Let $\mathbb{F}$ be the open cluster containing o. Then $V(\mathbb{F})$ is the set of vertices which can be reached from o via open bonds. For any $S \in \mathcal{A}_n$, a spanning tree on S has $|S|-1$ edges, and note that $|\partial S| \ge h|S|$ if $n = |\partial S|$ is large enough. Therefore

$$P(V(\mathbb{F}) = S) \ge p^{|S|-1}(1-p)^{|\partial S|} \ge p^{\frac{n}{h}-1}(1-p)^n.$$

Whence

$$1 \ge P(V(\mathbb{F}) \in \mathcal{A}_n) = \sum_{S \in \mathcal{A}_n} P(V(\mathbb{F}) = S) \ge |\mathcal{A}_n| p^{\frac{n}{h}-1}(1-p)^n.$$

Thus, if n is sufficiently large,

$$|\mathcal{A}_n| \le \left(\frac{1}{p}\right)^{\frac{n}{h}-1} \left(\frac{1}{1-p}\right)^n$$

holds for any p. Letting $p = 1/(1+h)$ concludes the proof. ∎

Proof of Theorem 1.1. Let $\mathbb{F}$ denote the open cluster containing o. Denote

$$\mathcal{A}_n(\mathbb{F}) = \{S \subset V(\mathbb{F})\,;\; o \in S,\; S \text{ is connected in } \mathbb{F}, |\partial_{\mathbb{G}} S| = n\},$$

where

$$\partial_{\mathbb{G}} S = \{e \in E(\mathbb{G}); e \text{ connects a vertex of } S \text{ to some point of } V(\mathbb{G}) \setminus S\}.$$

Suppose that $S \in \mathcal{A}_n(\mathbb{F})$. Then S is also a connected subset of $V(\mathbb{G})$. Similarly, let

$$\partial_{\mathbb{F}} S = \{e \in E(\mathbb{G}); e \text{ is open and connects a vertex of } S \text{ to some point of } V(\mathbb{F}) \setminus S\}.$$

For fixed $S \in \mathcal{A}_n$, each edge in $\partial_{\mathbb{G}} S$ is independently open with probability p. Thus

$$P\left(S \in \mathcal{A}_n(\mathbb{F}), \frac{|\partial_{\mathbb{F}} S|}{|\partial_{\mathbb{G}} S|} \le \alpha\right) \le P(Y_n \le \alpha n),$$

where Y_n has Binomial (n, p) law. By the Large Deviation Principle (see Dembo and Zeitouni (1998), Theorem 2.1.14, p.18), there is a rate function

$$I_p(\alpha) = \alpha \log \frac{\alpha(1-p)}{p(1-\alpha)} - \log \frac{1-p}{1-\alpha}$$

such that for $\alpha < p$ we have $I_p(\alpha) > 0$ and

$$P\left(\frac{|\partial_{\mathbb{F}} S|}{|\partial_{\mathbb{G}} S|} \leq \alpha\right) \leq e^{-I_p(\alpha)|\partial_{\mathbb{G}} S|}.$$

Recall $\Psi(h)$ defined in Lemma 2.1. When

$$p > 1 - 1/\Psi(h)$$

we have $I_p(0) = -\log(1-p) > \log \Psi(h)$. Thus there exists $\alpha_0 > 0$ such that $I_p(\alpha_0) > \log \Psi(h)$.

$$\begin{aligned}
& P\left(\exists S\,;\ \frac{|\partial_{\mathbb{F}} S|}{|\partial_{\mathbb{G}} S|} \leq \alpha_0,\ o \in S \subset V(\mathbb{G}),\ S \text{ is connected},\ k \leq |\partial_{\mathbb{G}} S|\right) \\
\leq & \sum_{n=k}^{\infty} \sum_{S \in \mathcal{A}_n} P\left(\frac{|\partial_{\mathbb{F}} S|}{|\partial_{\mathbb{G}} S|} \leq \alpha_0\right) \\
\leq & \sum_{n=k}^{\infty} \sum_{S \in \mathcal{A}_n} e^{-I_p(\alpha_0) n} = \sum_{k=n}^{\infty} |\mathcal{A}_n| e^{-I_p(\alpha_0) n} \\
\leq & \sum_{n=k}^{\infty} e^{-n(I_p(\alpha_0) - \log \Psi(h))} < C_1 e^{-C_2 k}
\end{aligned}$$

where C_1 and C_2 are positive constants. By the Borel-Cantelli Lemma,

$$\liminf_{n\to\infty} \left\{\frac{|\partial_{\mathbb{F}} S|}{|\partial_{\mathbb{G}} S|}\,;\ o \in S \subset V(\mathbb{G}),\ S \text{ is connected},\ n \leq |\partial_{\mathbb{G}} S|\right\} \geq \alpha_0 \quad a.s.$$

So

$$\liminf_{n\to\infty} \left\{\frac{|\partial_{\mathbb{F}} S|}{|S|}\,;\ o \in S \subset V(\mathbb{G}),\ S \text{ is connected},\ n \leq |S| < \infty\right\} \geq \alpha_0 \iota_E^*(\mathbb{G}).$$

∎

Proof of Theorem 1.2. If ν has an exponential tail, then there is a convex rate function $I(c) > 0$ such that $P(\sum_{i=1}^n L_i > cn) \leq \exp(-nI(c))$ for $c > EL_i$ and *i.i.d.* random variables $L_1, L_2, \cdots, L_n$ with distribution ν (see Dembo and Zeitouni (1998), Theorem 2.2.3, p.27.). Choose c large enough such that $I(c) > \log \Psi(h)$. For any $S \in \mathcal{A}_n$, let

$\mathsf{Edge}(S)$ be the set of edges with at least one end in S. Note that $\partial S \subset \mathsf{Edge}(S)$ and $|\partial S| \leq |\mathsf{Edge}(S)| \leq d|S|$ where d is an upper bound on degree in $\mathbb{G}$. Then

$$P\left(\frac{\sum_{e\in\mathsf{Edge}(S)} L_e}{d|S|} > c\right) \leq P\left(\frac{\sum_{i=1}^{d|S|} L_i}{d|S|} > c\right) \leq \exp(-d|S|I(c)) \leq \exp(-|\partial S|I(c)),$$

and

$$\begin{aligned}
&P\left(\exists S\,;\ \frac{\sum_{e\in\mathsf{Edge}(S)} L_e}{d|S|} > c, o \in S \subset V(\mathbb{G}),\ S \text{ is connected, } k \leq |\partial S|\right)\\
\leq &\sum_{n=k}^{\infty}\sum_{S\in\mathcal{A}_n} P\left(\frac{\sum_{e\in\mathsf{Edge}(S)} L_e}{d|S|} > c\right)\\
\leq &\sum_{n=k}^{\infty}\sum_{S\in\mathcal{A}_n} e^{-I(c)n} = \sum_{n=k}^{\infty} |\mathcal{A}_n| e^{-I(c)n}\\
\leq &\sum_{n=k}^{\infty} e^{-n(I(c)-\log\Psi(h))} < C_3 e^{-C_4 k}.
\end{aligned}$$

By the Borel-Cantalli Lemma, the probability of existence of a sequence $\{S_n\}$ such that

$$\frac{\sum_{e\in\mathsf{Edge}(S_n)} L_e}{d|S_n|} > c, o \in S_n \subset V(\mathbb{G}),\ S_n \text{ is connected, } \lim_n |\partial S_n| = \infty$$

is zero. In other words, for any sequence $\{S_n\}$ such that $o \in S_n \subset V(\mathbb{G})$, S_n is connected and $\lim_n |\partial S_n| = \infty$,

$$\frac{\sum_{e\in\mathsf{Edge}(S_n)} L_e}{d|S_n|} \leq c, \quad a.s.$$

Therefore

$$\lim_{n\to\infty} \inf\left\{\frac{|\partial S|}{\sum_{e\in\mathsf{Edge}(S)} L_e}\ ;\ o \in S \subset V(\mathbb{G}),\ S \text{ is connected, } n \leq |\partial S|\right\} \geq \frac{h}{dc} \quad a.s.$$

$\mathbb{G}^\nu$ is obtained from $\mathbb{G}$ by adding some new vertices. So $V(\mathbb{G})$ can be embeded into $V(\mathbb{G}^\nu)$ as a subset. In particular the basepoint of $\mathbb{G}^\nu$ corresponds to the basepoint of $\mathbb{G}$. For $o \in S \subset V(\mathbb{G})$, S is connected in $\mathbb{G}$, there is the unique *maximal* connected $\widetilde{S} \subset V(\mathbb{G}^\nu)$ such that $\widetilde{S} \cap V(\mathbb{G}) = S$. In computing $\iota_E^*(\mathbb{G}^\nu)$ it suffices to consider only these *maximal* $\widetilde{S}$'s, and $|\widetilde{S}| \leq \sum_{e\in\mathsf{Edge}(S)} L_e$. Consequently we conclude that $\iota_E^*(\mathbb{G}^\nu) \geq h/dc > 0$. ∎

REMARK 2.2. Suppose that the distribution of L does not have an exponential tail. Then for any $c > EL$ and any $\epsilon > 0$, we have $P(\sum_{i=1}^{n} L_i \geq 2cn) \geq e^{-\epsilon n}$ for sufficiently large n. Take a binary tree with root o as the basepoint. There are at least 2^n paths from level n to level $2n$ that do not intersect each other

$$\begin{aligned}
&P(\text{along at least one of these } 2^n \text{ paths } \sum_{i=1}^{n} L_i \geq 2cn) \\
\geq &1 - (1 - e^{-\epsilon n})^{2^n} \geq 1 - \exp(-e^{-\epsilon n} 2^n) \to 1.
\end{aligned}$$

With probability very close to 1 (depending on n) there is a path from level n to $2n$ along which $\sum_{i=1}^{n} L_i \geq 2cn$. Take the path and extend it to the root o. Let S be the set of vertices in the extended path from the root o to level $2n$. Then

$$\frac{|\partial S|}{\sum_{e \in \mathsf{Edge}(S)} L_e} \leq \frac{2(2n+1)}{2cn} \approx \frac{2}{c}.$$

Since c can be chosen large, $\iota_E^*(\mathbb{G}^\nu) = 0$. This shows that the *exponential tail* condition is necessary to ensure the positivity of $\iota_E^*(\mathbb{G}^\nu)$.

Proof of Corollary 1.3. A Galton-Watson process is uniquely determined by the offspring distribution $\{p_0, p_1, p_2, \cdots, p_k, \cdots\}$. Let $\mathbb{T}$ be a Galton-Watson tree and o its root.

Case i). $p_0 = p_1 = 0$. For any finite $S \subset V(\mathbb{T})$, $|S| \leq |\partial S|(\frac{1}{2} + \frac{1}{2^2} + \cdots) \leq |\partial S|$. So $\iota_E^*(\mathbb{T}) \geq \iota_E(\mathbb{T}) \geq 1$.

Case ii). $p_0 = 0$, $p_1 > 0$. In this case the Galton-Watson tree $\mathbb{T}$ can be viewed as random stretch $\mathbb{G}^\nu$ of another Galton-Watson tree $\mathbb{G}$, where $\mathbb{G}$ is generated according to $p'_k = p_k/(1-p_1)$, $k = 2, 3, \cdots$, $p'_0 = p'_1 = 0$ and ν is the geometric distribution with parameter p_1. By Theorem 1.2, $\iota_E^*(\mathbb{T}) = \iota_E^*(\mathbb{G}^\nu) > 0$ *a.s.*

Case iii). $p_0 > 0$, $p_0 + p_1 < 1$ and $\sum k p_k > 1$. Let $f(s) = \sum_{i=0}^{\infty} p_k s^k$ and let $q < 1$ be the extinction probability, *i.e*, $q = f(q)$. An infinite Galton-Watson tree can be constructed as follows, see Lyons (1992). Begin with the root which is declared to be *open*. Add to the root a random number of edges according to probability distribution $P(Y = k) = p_k(1 - q^k)/(1 - q)$. Declare each vertex *open* with probability $1 - q$ and *closed* with probability q, independent of each other. If all the newly added vertices are *closed*, discard the entire assignment and reassign *open/closed* all over again. For each *open* vertex, repeat the same procedure. To each *closed* vertex, attach to it independently a Galton-Watson tree conditioned to be finite.

The subtree $\mathbb{T}_1$ consisting of *open* vertices and edges connecting them is a Galton-Watson tree without leaves, and $\iota_E^*(\mathbb{T}_1) > 0$ according to part ii). For each *open* vertex x

of $\mathbb{G}$, label its offspring from 1 to Y_x, where Y_x is a random variable with $P(Y_x = k) = p_k(1-q^k)/(1-q)$. Along the sequence of Y_x vertices, each is *open* with probability $1-q$ and *closed* with probability q (independent of each other if we ignore the constraint that there is at least one *open* vertex). The number of *closed* vertices before the first *open* vertex is stochastically bounded above by a random variable with a geometric distribution. The same statement holds for the number of *closed* vertices after the last *open* vertex, and for the number of *closed* vertices between the k-th *open* vertex and $(k+1)$-th *open* vertex.

Let L_1 be the total number of vertices of finite Galton-Watson trees attached to the *closed* vertices before the second *open* vertex (if it ever exists). Similarly, let L_2 be the total number of vertices of finite Galton-Watson trees attached to the *closed* vertices between the second *open* vertex and the third *open* vertex (if it ever exists). And so on, until the last *open* vertex among the offspring of x. The variables $L_2, L_3, \cdots$ are *i.i.d.*; L_1 is independent of other L_i's but has a different distribution. Thus we may identify the Galton-Watson tree $\mathbb{T}$ as a random stretch of $\mathbb{T}_1$ in computing $\iota_E^*(\mathbb{T})$. Although there are two different distributions in the random stretch, the same argument works since both have exponential tails.

All L_i's are stochastically dominated by $\sum_{j=1}^{W_1+W_2} U_j$, where $W_1, W_2, U_1, U_2, \cdots$ are random variables, independent of each other, $P(W_i = k) = q^k(1-q)$, $k = 0, 1, 2, \cdots$, and U_j is the size of a Galton-Watson tree conditioned on extinction. Let ν be the probability distribution of $\sum_{j=1}^{W_1+W_2} U_j$. By the next lemma we conclude that ν has an exponential tail. Applying Theorem 1.2 completes the proof. ∎

LEMMA 2.3. *(Theorem 13.1, Harris(1963)) For a supercritical Galton-Watson process, the size of a Galton-Watson tree conditioned on extinction has a distribution with an exponential tail.*

§3. Speed of Random Walk.

Let $\mathbb{G}$ be the Cayley graph of a finitely generated group G with a fixed set of generators. We identify vertices of $\mathbb{G}$ with elements of the group G. Two points x and y of $\mathbb{G}$ are neighbors if xy^{-1} or yx^{-1} is a generator.

Let $\mathbb{F}$ be the Cayley graph of a finite group F generated by a fixed set of generators.

An element of $\sum_{x\in\mathbb{G}_N} \mathbb{F}$ is called a *configuration* and is denoted by $\eta = \{\eta(x)\,;\, x \in V(\mathbb{G})\}$, where $\eta(x) \in V(\mathbb{F})$ is the x-coordinate of η. In the following discussion, we only consider those η's such that $\eta(x)$ is the unit element of F for all but finite many x's.

Define a new graph $\mathbb{G} \ltimes \sum_{x\in\mathbb{G}} \mathbb{F}$ as a semidirect product of $\mathbb{G}$ with the direct sum of copies of $\mathbb{F}$ indexed by $\mathbb{G}$. Vertices of $\mathbb{G} \ltimes \sum_{x\in\mathbb{G}} \mathbb{F}$ are identified as $\{(m,\eta)\,;\, m \in V(\mathbb{G}), \eta \in$

$\sum_{x\in\mathbb{G}}\mathbb{F}\}$. Two vertices, (m,η) and (m_1,ξ), are neighbors if either (i) $m=m_1$, $\eta(x)=\xi(x)$ for all $x\neq m$, $\eta(m)(\xi(m))^{-1}$ or $\xi(m)(\eta(m))^{-1}$ is a generator of F; or (ii) $\eta=\xi$, mm_1^{-1} or m_1m^{-1} is a generator of G. In particular, if $\mathsf{F}=\{0,1\}$ is the group of two elements and $\mathbb{G}$ is $\mathbb{Z}^d$, then $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$ is exactly $\mathbb{G}_d$ discussed before Theorem 1.5.

Suppose that $\mathbb{G}$ is amenable and that $\mathbb{F}$ is the Cayley graph of a finite group. Then the graph $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$ is amenable and grows exponentially. By Burton and Keane (1989), there is only one infinite cluster when percolation occurs. Let X be the simple random walk in the unique infinite cluster ω. By Lemma 4.2 of BLS (1999), the speed of random walk X exists.

We say $\mathbb{G}$ is *recurrent* if the simple random walk in $\mathbb{G}$ is recurrent, Namely, G is a finite extension of $\mathbb{Z}^1$ or $\mathbb{Z}^2$. (see, e.g. Woess (2000), Theorem 3.24, p.36). The following theorem is a generalization of Theorem 1.5.

THEOREM 3.1. *Suppose that $\mathbb{G}$ is a recurrent Cayley graph and that $\mathbb{F}$ is the Cayley graph of a finite group. Then the simple random walk in the infinite cluster of supercritical Bernoulli bond percolation in $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$ has zero speed a.s. Namely,* $\lim_n \frac{1}{n}|X_n|_\omega=0$ *a.s.*

On the other hand, if $\mathbb{G}$ is transient, then for p sufficiently close to 1, the infinite cluster of p-Bernoulli bond percolation in $\mathbb{G}$ is transient. (Benjamini & Schramm (1998), also see Theorem 9 in Angel, Benjamini, Berger & Peres (2002)). In particular, if $d\geq 3$ and $\mathbb{G}=\mathbb{Z}^d$, this is true for all $p>p_c(\mathbb{Z}^d)$. (Grimmett, Kesten & Zhang (1993)). The following theorem is a generalization of Theorem 1.6.

THEOREM 3.2. *Suppose that the infinite cluster of p-Bernoulli bond percolation in the Cayley graph $\mathbb{G}$ is transient and that $\mathbb{F}$ is the Cayley graph of a finite group. Then the simple random walk in the infinite cluster of p-Bernoulli bond percolation in $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$ has positive speed a.s. Namely,* $\lim_n \frac{1}{n}|X_n|_\omega>0$ *a.s.*

Fix a vertex o of $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$ as the basepoint, *e.g.*, the vertex corresponding to the unit element of the group. Let $\|x\|$ be the distance between vertex x and the basepoint o in $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$. Certainly, $\|x\|\leq|x|_\omega$. However, according to Lemma 4.6 of BLS (1999), $\lim_n\|X_n\|/n=0$ implies that $\lim_n|X_n|_\omega/n=0$. For this reason we shall consider $\|x\|$ instead of $|x|_\omega$.

It will be useful to consider **delayed simple random walk** $Z=Z^\omega$ on ω, defined as follows. Set $Z(0)$ be some fixed vertex of $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$. If $n\geq 0$, conditioned on $\langle Z(0),\ldots,Z(n)\rangle$ and ω, let $Z'(n+1)$ be chosen from $Z(n)$ and its neighbors in $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$ with equal probability. Set $Z(n+1):=Z'(n+1)$ if the edge $[Z(n),Z'(n+1)]$ belongs to

ω; otherwise, let $Z(n+1) := Z(n)$. By Lemma 4.2 of BLS (1999), the speed of delayed random walk Z exists.

LEMMA 3.3.

$$\lim_{n\to\infty} \frac{\|X_n\|}{n} \geq \lim_{n\to\infty} \frac{\|Z(n)\|}{n} \geq c \lim_{n\to\infty} \frac{\|X_n\|}{n}, \tag{3.1}$$

where $c > 0$ is a deterministic constant.

Proof. Compare $\langle Z(0), Z(1), \cdots, Z(n), \cdots\rangle$ with the random walk $\langle X_0, X_1, \cdots, X_n, \cdots\rangle$. A typical sample path of Z is obtained from a sample path of X by repeating X_n a random number of times, with a geometric distribution. The parameter of the geometric distribution depends on X_n and ω but is uniformly bounded above by $d/(d+1)$ and below by $1/(d+1)$, where d is the degree of a vertex of the graph $\mathbb{G} \ltimes \sum_{x\in\mathbb{G}} \mathbb{F}$. Therefore (3.1) holds. ∎

Let Z be the delayed random walk in a cluster ω. Denote by P_ω and E_ω the probability distribution and expectation of Z for fixed ω. Denote by $\mathbf{E}$ the average over all realizations of ω. Write $Z(n) = (m_n, \eta_n)$ and call the first component m_n the marker. A key observation is that the motion of the marker is recurrent in the following sense.

LEMMA 3.4. *Suppose that $\mathbb{G}$ is a recurrent Cayley graph and that $\mathbb{F}$ is the Cayley graph of a finite group. Let Z be the delayed random walk and write $Z(n) = (m_n, \eta_n)$. Then*

$$\mathbf{E}P_\omega(m_n = m_0 \textit{ for some } n \geq 1) = 1. \tag{3.2}$$

Proof. Let $|x|_{\mathbb{G}}$ be the distance between $x \in V(\mathbb{G})$ and the basepoint o in $\mathbb{G}$, and N a large integer. Introduce two stopping times:

$$\tau_N = \min\{n \geq 0; |m_n|_{\mathbb{G}} = N\}$$
$$\text{and } \tau_o^+ = \min\{n \geq 1; m_n = o\}.$$

Then (3.2) can be rewritten as

$$\lim_{N\to\infty} \mathbf{E}P_\omega(\tau_N < \tau_o^+ | m_0 = o) = 0.$$

$P_\omega(\tau_N < \tau_o^+|m_0 = o)$ may be approximated by the counterpart in a subset of $\mathbb{G} \ltimes \sum_{x\in\mathbb{G}} \mathbb{F}$.

Let $\mathbb{G}_N = \{x \in \mathbb{G};\ |x|_{\mathbb{G}} \leq N\}$. Define $\mathbb{G}_N \ltimes \sum_{x\in\mathbb{G}_N} \mathbb{F}$ as $\mathbb{G} \ltimes \sum_{x\in\mathbb{G}} \mathbb{F}$ is defined. There may be several disjoint clusters in a realization of p-Bernoulli bond percolation in $\mathbb{G}_N \ltimes \sum_{x\in\mathbb{G}_N} \mathbb{F}$ and each cluster may have several vertices with the marker at o. Take a cluster with at least one vertex whose marker is at o, say $\mathbb{H}$. If there are k vertices in $\mathbb{H}$ with the marker at o, "glue" these k vertices together as one vertex denoted by Θ. Let

$\mathbb{H}'$ be the modified graph of the cluster $\mathbb{H}$. Coupling the delayed simple random walks in $V(\mathbb{H})$ and in $V(\mathbb{H}')$, we get the following equation.

$$\frac{1}{k}\sum_{x\in V(\mathbb{H}),m(x)=o} P_\omega(\tau_N<\tau_o^+|Z(0)=x)=P_\omega(\tilde{\tau}_N<\tilde{\tau}_o^+|Z'(0)=\Theta) \tag{3.3}$$

where

$$\tilde{\tau}_N=\min\{n\geq 0;|m_n|_{\mathbb{G}}=N\}\quad\text{ and }\quad\tilde{\tau}_o^+=\min\{n\geq 1;m_n=o\}$$

are the stopping time for the delayed simple random walk Z' in $V(\mathbb{H}')$.

The delayed simple random walk Z' in $V(\mathbb{H}')$ is a reversible Markov chain with respect to the measure π where $\pi(\Theta)=k$, and $\pi(x)=1$ for all other $x\in V(\mathbb{H}')$, $x\neq\Theta$. Applying the Dirichlet Principle (see Liggett (1985), p.99), we have that

$$\begin{aligned}2\pi(\Theta)P_\omega(\tilde{\tau}_N<\tilde{\tau}_o^+|Z(0)=\Theta)&=\inf_{f\in\mathcal{F}}\sum_{u\in V(\mathbb{H}')}\sum_{[u,v]\in E(\mathbb{H}')}\pi(u)p(u,v)(f(u)-f(v))^2\\&=\inf_{f\in\mathcal{F}}\sum_{[u,v]\in E(\mathbb{H}')}\frac{2}{d+1}(f(u)-f(v))^2\\&=\inf_{f\in\mathcal{F}}\sum_{[u,v]\in E(\mathbb{H})}\frac{2}{d+1}(f(u)-f(v))^2\end{aligned} \tag{3.4}$$

where $\mathcal{F}$ is the class of functions with the following properties.

$$f:V(\mathbb{H})\cup\{\Theta\}\to[0,1];\quad f(x)=0\text{ if }x=\Theta\text{ or }m(x)=o;\quad f(x)=1\text{ if }|m(x)|_{\mathbb{G}}=N. \tag{3.5}$$

In particular, let $\{Y_n\}$ be the simple random walk in $\mathbb{G}$ and

$$\begin{aligned}&\sigma_N=\min\{n\geq 0;|Y_n|_{\mathbb{G}}=N\};\\&\sigma_o^+=\min\{n\geq 1;Y_n=o\};\\&\rho(m)=P(\sigma_N<\sigma_o^+|Y_0=m).\end{aligned}$$

Then $f(x)=\rho(m(x))$ is in $\mathcal{F}$. Plugging it into (3.4), in light of equation (3.3), we conclude that

$$\sum_{x\in V(\mathbb{H}),m(x)=o}P_\omega(\tau_N<\tau_o^+|Z(0)=x)\leq\frac{1}{d+1}\sum_{[u,v]\in E(\mathbb{H})}(\rho(m(u))-\rho(m(v)))^2.$$

Notice that $P_\omega(\tau_N<\tau_o^+|Z(0)=x)=0$ if $m(x)=o$ and there is no y in the cluster such that $|m(y)|_{\mathbb{G}}=N$. Summing over all disjoint clusters, we get

$$\begin{aligned}\sum_{x:m(x)=o}P_\omega(\tau_N<\tau_o^+|Z(0)=x)&=\sum_{\mathbb{H}}\sum_{x\in V(\mathbb{H}),m(x)=o}P_\omega(\tau_N<\tau_o^+|Z(0)=x)\\&\leq\frac{|\mathbb{F}|^{|\mathbb{G}_N|}}{d+1}\sum_{[u,v]\in E(\mathbb{G}_N)}(\rho(u)-\rho(v))^2.\end{aligned}$$

Taking the average over all realizations of percolation in $\mathbb{G}_N \ltimes \sum_{x\in\mathbb{G}_N} \mathbb{F}$, then $\mathbf{E}P_\omega(\tau_N < \tau_o^+|Z(0)=(o,\eta))$ is independent of η. There are $|\mathbb{F}|^{|\mathbb{G}_N|}$ vertices in $\mathbb{G}_N \ltimes \sum_{x\in V(\mathbb{G}_N)} \mathbb{F}$ with the marker at o.

$$|\mathbb{F}|^{|\mathbb{G}_N|}\mathbf{E}P_\omega\left(\tau_N < \tau_o^+|Z(0)=(o,\eta)\right) \le \frac{|\mathbb{F}|^{|\mathbb{G}_N|}}{d+1} \sum_{[u,v]\in E(\mathbb{G}_N)} (\rho(u)-\rho(v))^2.$$

After cancellation,

$$\begin{aligned}\mathbf{E}P_\omega(\tau_N < \tau_o^+|Z(0)=(o,\eta)) &\le \frac{1}{d+1} \sum_{[u,v]\in E(\mathbb{G}_N)} (\rho(u)-\rho(v))^2 \\ &= \frac{1}{d+1} P(\sigma_N < \sigma_o^+|Y_0=m) \to 0 \text{ as } N\to\infty,\end{aligned}$$

since the simple random walk in $\mathbb{G}$ is recurrent. ∎

Proof of Theorem 3.1. Let Z be the delayed simple random walk in the infinite cluster. By Lemma 3.3, it suffices to show that $\lim_n \|Z(n)\|/n = 0$. a.s.

Let $\mathcal{R}_n = \{m_0, m_1, \cdots, m_n\} \subset V(\mathbb{G})$ be the range of the marker up to time n. Then

$$\begin{aligned}|\mathcal{R}_n| =&1 + \sum_{k=0}^{n-1} 1_{\{m_k\ne m_{k+1}, m_k\ne m_{k+2},\cdots,m_k\ne m_n\}} \\ \le&\ell + \sum_{k=0}^{n-l} 1_{\{m_k\ne m_{k+1}, m_k\ne m_{k+2},\cdots,m_k\ne m_{k+\ell}\}}\end{aligned}$$

for some fixed integer ℓ. Since the cluster is randomly generated, consider (ω, Z) in an enlarged probability space in which the infinite cluster ω seen from the walker $Z(n)$ is stationary distributed. Then

$$\{1_{\{m_k\ne m_{k+1}, m_k\ne m_{k+2},\cdots,m_k\ne m_{k+\ell}\}}; k=0,1,2,3,\cdots\}$$

is a stationary sequence in the enlarged space.

$$\begin{aligned}\mathbf{E}E_\omega \limsup_n \frac{|\mathcal{R}_n|}{n} &\le \mathbf{E}E_\omega \lim_n \frac{1}{n}\sum_{k=0}^{n-l} 1_{\{m_k\ne m_{k+1}, m_k\ne m_{k+2},\cdots,m_k\ne m_{k+\ell}\}} \\ &= \mathbf{E}P_\omega(m_0\ne m_1, m_0\ne m_2,\cdots, m_0\ne m_\ell) \to 0 \qquad (3.6)\end{aligned}$$

as $\ell\to\infty$, by the Ergodic Theorem and by Lemma 3.4. Notice that $\mathcal{R}_n$ is connected in $V(\mathbb{G})$, and all sites can be visited within $2|\mathcal{R}_n|$ steps at most by the depth-first search strategy along a spanning tree in $\mathcal{R}_n$. Therefore in $\mathbb{G}\ltimes\sum_{x\in\mathbb{G}}\mathbb{F}$,

$$\|Z(n)\| \le |m_n|_{\mathbb{G}} + 2|\mathcal{R}_n| + \sum_{x\in\mathcal{R}_n} |\eta_n(x)|_{\mathbb{F}} \le (1+2+|\mathbb{F}|)|\mathcal{R}_n|.$$

We conclude from (3.6) that $\mathbf{E}E_\omega \limsup_{n\to\infty}\|Z(n)\|/n = 0$. So $\lim_{n\to\infty}\|Z(n)\|/n = 0$ *a.s.* ∎

LEMMA 3.5. *Suppose that $\mathbb{G}$ is an infinite Cayley graph and $\mathbb{F}$ is the Cayley graph of a finite group. Suppose that the infinite cluster of p-Bernoulli bond percolation on $\mathbb{G}$ is transient. Let Z be the delayed random walk in the infinite cluster of p-Bernoulli bond percolation on $\mathbb{G} \ltimes \sum_{x\in\mathbb{G}} \mathbb{F}$ and write $Z(n) = (m_n, \eta_n)$. Then*

$$\mathbf{E}P_\omega(m_n \neq m_0 \text{ for all } n \geq 1) > 0.$$

Proof. We shall prove

$$\lim_{N\to\infty} \mathbf{E}P_\omega(\tau_N < \tau_o^+ | m_0 = o) > 0, \tag{3.7}$$

where τ_N and τ_o^+ are stopping times defined in the proof of Lemma 3.4.

Recall the finite graphs $\mathbb{G}_N$ and $\mathbb{G}_N \ltimes \sum_{x\in\mathbb{G}_N} \mathbb{F}$ defined in the proof of Lemma 3.4. Vertices (m, η) of $\mathbb{G}_N \ltimes \sum_{x\in\mathbb{G}_N} \mathbb{F}$ are classified into $|\mathbb{F}|^{|\mathbb{G}_N|}$ classes according to the second component η. For fixed configuration η_1, denote by $\mathbb{G}_N(\eta_1)$ the subgraph induced by the class of vertices $\{(m, \eta_1); m \in V(\mathbb{G}_N)\}$. Clearly, $\mathbb{G}_N(\eta_1)$ is isomorphic to $\mathbb{G}_N$ for any $\eta_1 \in \sum_{x\in\mathbb{G}_N} \mathbb{F}$. Let the cluster within $\mathbb{G}_N(\eta_1)$ containing (o, η_1) be

$$C_o(\eta_1) = \{(m, \eta_1);\ \ (m, \eta_1) \leftrightarrow (o, \eta_1) \text{ within } \mathbb{G}_N(\eta_1)\}.$$

Run a simple random walk Y in $C_o(\eta_1)$ starting from (o, η_1). Recall $\sigma_N = \min\{n \geq 0; |Y_n|_{\mathbb{G}} = N\}$ and $\sigma_o^+ = \min\{n \geq 1; Y_n = o\}$. Then $P(\sigma_N < \sigma_o^+ | Y_0 = o)$ is decreasing in N. The hypothesis of the lemma (transience of the infinite cluster) means that

$$\lim_{N\to\infty} \mathbf{E}P_\omega(\sigma_N < \sigma_o^+ | Y_0 = o) > 0. \tag{3.8}$$

There may be several disjoint clusters in a realization of p-Bernoulli bond percolation in $\mathbb{G}_N \ltimes \sum_{x\in\mathbb{G}_N} \mathbb{F}$ and each cluster may have several vertices with the marker at o. Take a cluster, say $\mathbb{H}$, and run the delayed simple random walk Z in $V(\mathbb{H})$. It follows from (3.3) and (3.4) that

$$\sum_{x\in V(\mathbb{H}), m(x)=o} P_\omega(\tau_N < \tau_o^+ | Z(0) = x) = \inf_{f\in\mathcal{F}} \sum_{[u,v]\in E(\mathbb{H})} \frac{1}{d_{\mathbb{G}} + d_F + 1}(f(u) - f(v))^2, \tag{3.9}$$

where $\mathcal{F}$ is the class of functions satisfying (3.5), $d_{\mathbb{G}}$ and $d_{\mathbb{F}}$ are the degrees of a vertex of $\mathbb{G}$ and $\mathbb{F}$ respectively. Notice that (3.9) is still valid even there is no vertex y such that $|m(y)|_G = N$. Summing over all disjoint clusters, we get

$$\begin{aligned}\sum_\eta P_\omega(\tau_N < \tau_o^+ | Z(0) = (o, \eta)) &= \sum_{\mathbb{H}} \sum_{(o,\eta)\in V(\mathbb{H})} P_\omega(\tau_N < \tau_o^+ | Z(0) = (o, \eta)) \\ &= \inf_{f\in\mathcal{F}} \sum \frac{1}{d_{\mathbb{G}} + d_{\mathbb{F}} + 1}(f(u) - f(v))^2, \end{aligned} \tag{3.10}$$

where the summation is over all open bonds $[u,v]$ of $\mathbb{G}_N \ltimes \sum_{x\in\mathbb{G}_N}\mathbb{F}$. Every term in (3.10) is non-negative. Discarding those terms involving open edge $[u,v]$ where u and v are in different classes (*i.e.* markers of u and v are the same), we get the following inequality.

$$\begin{aligned}\text{RHS of (3.10)} \geq & \inf_{f\in\mathcal{F}}\sum_{\eta}\sum_{[u,v]\in E(\mathbb{G}_N(\eta)),\text{ open}}\frac{1}{d_{\mathbb{G}}+d_{\mathbb{F}}+1}(f(u)-f(v))^2\\ =&\frac{d_{\mathbb{G}}+1}{d_{\mathbb{G}}+d_{\mathbb{F}}+1}\sum_{\eta}P(\sigma_N<\sigma_o^+|Y_0=o). \end{aligned}\tag{3.11}$$

Combining (3.11) with (3.10), we conclude that for every realization ω of the Bernoulli bond percolation,

$$\sum_{\eta}P_\omega(\tau_N<\tau_o^+|Z(0)=(o,\eta))\geq\sum_{\eta}\frac{d_{\mathbb{G}}+1}{d_{\mathbb{G}}+d_{\mathbb{F}}+1}P_\omega(\sigma_N<\sigma_o^+|Y_0=o).\tag{3.12}$$

Taking expectation over all realizations ω of the Bernoulli bond percolation, then $\mathbf{E}P_\omega(\tau_N < \tau_o^+|Z(0)=(o,\eta))$ is independent of η. It follows from (3.12) that

$$\mathbf{E}P_\omega(\tau_N<\tau_o^+|Z(0)=(o,\eta))\geq\frac{d_{\mathbb{G}}+1}{d_{\mathbb{G}}+d_{\mathbb{F}}+1}\mathbf{E}P_\omega(\sigma_N<\sigma_o^+|Y_0=o).$$

Taking the limit as $N\to\infty$, inequality (3.7) then follows from (3.8). ∎

Proof of Theorem 3.2. Consider the **delayed simple random walk** $Z = Z^\omega$ on the infinite cluster ω. The delayed simple random walk Z in ω is a reversible Markov chain with respect to the uniform measure. Therefore

$$\mathbf{E}P_\omega(m_i\neq m_n, 0\leq i\leq n-1)=\mathbf{E}P_\omega(m_i\neq m_0, 1\leq i\leq n).\tag{3.13}$$

Define

$$\begin{aligned}\zeta(k)=&1\quad\text{if }\eta_k\neq\eta_{k-1}, m_i\neq m_k\text{ for }0\leq i\leq k-2\text{ and for }i\geq k+1;\\ =&0\quad\text{otherwise.}\end{aligned}$$

Then

$$\begin{aligned}&E_\omega\zeta(k)=P_\omega(\eta_k\neq\eta_{k-1}\text{ and }m_i\neq m_k\text{ for }0\leq i\leq k-2\text{ and for }i\geq k+1)\\ =&P_\omega(m_i\neq m_k\text{ for all }i\geq k+1|\eta_k\neq\eta_{k-1}\text{ and }m_i\neq m_k\text{ for }0\leq i\leq k-2)\\ &P_\omega(\eta_k\neq\eta_{k-1}|m_i\neq m_{k-1}\text{ for }0\leq i\leq k-2)\\ &P_\omega(m_i\neq m_k\text{ for }i=0,1,2,\cdots,k-2).\end{aligned}$$

In the big probability space, the distribution of (Z, ω) is invariant under the shift by Lemma 4.1 of BLS (1999). Taking the expectation over all possible realizations of ω, then

$$\begin{aligned}
&\mathbf{E}P_\omega(m_i \neq m_k \text{ for all } i \geq k+1 | \eta_k \neq \eta_{k-1} \text{ and } m_i \neq m_k \text{ for } 0 \leq i \leq k-2) \\
=&\mathbf{E}P_\omega(m_i \neq m_0 \text{ for all } \geq 1).
\end{aligned}$$

Moreover,

$$\mathbf{E}P_\omega(\eta_k \neq \eta_{k-1} | m_i \neq m_{k-1} \text{ for } 0 \leq i \leq k-2) = \frac{pd_{\mathbb{F}}}{d_{\mathbb{G}} + d_{\mathbb{F}} + 1};$$

and using reversibility (3.13),

$$\mathbf{E}P_\omega(m_i \neq m_{k-1} \text{ for } 0 \leq i \leq k-2) \geq \mathbf{E}P_\omega(m_i \neq m_0 \text{ for all } i \geq 1).$$

Therefore

$$\mathbf{E}E_\omega \zeta(k) \geq \frac{pd_{\mathbb{F}}}{d_{\mathbb{G}} + d_{\mathbb{F}} + 1}(\mathbf{E}P_\omega(m_i \neq m_0 \text{ for all } i \geq 1))^2.$$

Finally, because $\|Z(n)\| \geq \sum_{k=1}^n \zeta(k)$,

$$\begin{aligned}
&\mathbf{E}E_\omega \lim_n \frac{|Z(n)|_\omega}{n} \geq \mathbf{E}E_\omega \lim_n \frac{\|Z(n)\|}{n} = \lim_n \mathbf{E}E_\omega \frac{\|Z(n)\|}{n} \\
&\geq \lim_n \mathbf{E}E_\omega \frac{1}{n} \sum_{k=1}^n \zeta(k) \geq \lim_n \frac{1}{n} \sum_{k=1}^n \mathbf{E}E_\omega \zeta(k) > 0.
\end{aligned}$$

Since $\lim_n \frac{1}{n}|Z(n)|_\omega$ exists and is a constant *a.s.*, it must be positive. By Lemma 3.3, the speed $\lim_n \frac{1}{n}|X_n|_\omega$ must be also positive. ∎

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